Chapter 4

Application of Tensors in Special Relativity

4.1 The energy - momentum tensor

Consider a pressure - less distribution of non - interacting particles [called dust], with rest mass m and number density n in the momentarily comoving reference frame [MCRF] .

The density in this frame is

$$\rho = nm \ . \tag{4.1}$$

In a general frame the number density will go up by a factor γ

$$n \rightarrow \gamma n \tag{4.2}$$

and

$$m \rightarrow \gamma m \ , \tag{4.3}$$

so

$$\rho \rightarrow \left(1 - \tfrac{v^2}{c^2}\right)^{-1} \rho \ . \tag{4.4}$$

Thus the density ρ is not a component of a four - vector. We will see that it is a component of a 2/0 tensor.

We can introduce a number flux four - vector $\mathbf{N}$:

$$\mathbf{N} = n\mathbf{U} \rightarrow_{\mathcal{O}} \gamma\,(nc, n\mathbf{v}) = \gamma\,(nc, nv^x, nv^y, nv^z) \ \ , \tag{4.5}$$

where $\gamma n v^x$ is the flux per unit area across a surface with normals in the x direction etc, and γn can be interpreted as the flux across a constant t surface. Thus $\mathbf{N}$ combines the flux and the number density in a single four-dimensional quantity. Note that

$$\mathbf{N} \cdot \mathbf{N} = -n^2 c^2 \; . \tag{4.6}$$

The most convenient definition of the energy-momentum tensor is in terms of its components in some arbitrary frame.

$$\mathbf{T}\left(\tilde{\mathbf{d}}x^\alpha, \tilde{\mathbf{d}}x^\beta\right) = T^{\alpha\beta} \; , \tag{4.7}$$

where $T^{\alpha\beta}$ is the flux of α-momentum across a surface of constant x^β. By α-momentum we mean the α component of the $four$-momentum p^α.

Let us see how this definition fits in with the discussion above. Consider first T^{00}. This is defined as the flux of 0-momentum [energy divided by c] across a surface of constant t. This is just the energy density.

$$T^{00} = energy\ density \; . \tag{4.8}$$

Similarly, T^{0i} is the flux of energy divided by c across a surface of constant x^i:

$$T^{0i} = (energy\ flux/c)\ \ across\ a\ surface\ x^i = const \; . \tag{4.9}$$

Then T^{i0} is the flux of i-momentum across a surface of constant t: the density of i-momentum multiplied by c:

$$T^{i0} = i - momentum\ density \times c \; . \tag{4.10}$$

Finally T^{ij} is the j-flux of i-momentum:

$$T^{ij} = flux\ of\ \ i - momentum\ across\ a\ surface\ x^j \; . \tag{4.11}$$

For any particular system, giving the components of $\mathbf{T}$ in some frame, defines it completely.

For dust, the components of T in the MCRF are particularly simple. There is no motion of the particles, so all i-momenta are zero and all spatial fluxes are zero. Therefore:

$$
\begin{aligned}
\left(T^{00}\right)_{MCRF} &= \rho c^2 = mnc^2 , \\
\left(T^{0i}\right)_{MCRF} &= \left(T^{i0}\right)_{MCRF} = \left(T^{ij}\right)_{MCRF} = 0 .
\end{aligned} \tag{4.12}
$$

It is easy to see that the tensor $\mathbf{P}\otimes\mathbf{N}$ has exactly these components in the MCRF, where $\mathbf{P} = m\mathbf{U}$ is the four-momentum of a particle. It follows that, for dust we have

$$
\mathbf{T} = \mathbf{P} \otimes \mathbf{N} = mn\mathbf{U} \otimes \mathbf{U} = \rho\mathbf{U} \otimes \mathbf{U} . \tag{4.13}
$$

From this we conclude that the components of $\mathbf{T}$ are:

$$
T^{\alpha\beta} = \rho U^{\alpha} U^{\beta} , \tag{4.14}
$$

or in matrix form:

$$
T^{\alpha\beta} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{4.15}
$$

In a frame $\mathcal{O}$ with $\mathbf{U} \to_{\bar{\mathcal{O}}} \gamma\,(c, \mathbf{v})$, we therefore have

$$
\begin{aligned}
T^{\bar{0}\bar{0}} &= \rho U^{\bar{0}} U^{\bar{0}} = \gamma^2 \rho c^2 , \\
T^{\bar{0}\bar{i}} &= \rho U^{\bar{0}} U^{\bar{i}} = \gamma^2 \rho c v^{\bar{i}} , \\
T^{\bar{i}\bar{0}} &= \rho U^{\bar{i}} U^{\bar{0}} = \gamma^2 \rho c v^{\bar{i}} , \\
T^{\bar{i}\bar{j}} &= \rho U^{\bar{i}} U^{\bar{j}} = \gamma^2 \rho v^{\bar{i}} v^{\bar{j}} .
\end{aligned} \tag{4.16}
$$

These are exactly what we would calculate from first principles, for the energy density, energy flux, momentum density and momentum flux respectively. Notice one important property of $\mathbf{T}$: it is symmetric:

$$
T^{\alpha\beta} = T^{\beta\alpha} . \tag{4.17}
$$

This will turn out to be true in general, not just for dust.

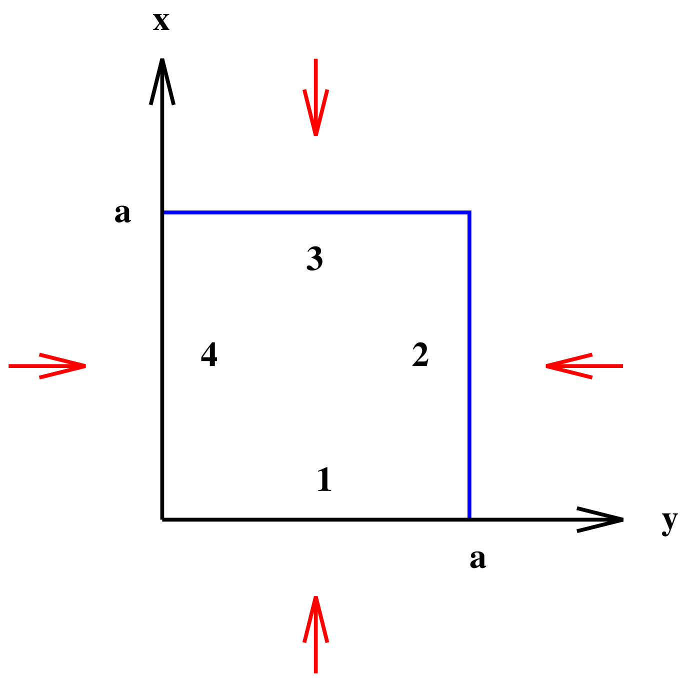

Figure 4.1: Energy flow across a fluid element.

4.1.1 General fluids

A general fluid has pressure p [random velocities] and viscosity [interaction between particles]. $T^{\alpha\beta}$ has the same interpretion as for dust except that T^{0i} and T^{ij} need no longer be zero in the MCRF. In particular T^{0i} represents heat conduction and T^{ij} represents the stress forces between adjacent fluid elements [i'th component of force per area across a surface of constant x^j].

4.1.2 Conservation of energy - momentum

Since **T** represents the energy and momentum content of the fluid, there must be some way of using it to express the law of conservation of energy and momentum. In fact it is reasonably easy.

Consider a cubical fluid element [see Figure 4.1] of side a, seen only in cross section below [z direction suppressed]. Energy can flow across all sides. The rate of flow across face (4) is $ca^2T^{0x}(x = 0)$, and across (2) is $-ca^2T^{0x}(x = a)$; the second term has a minus sign because T^{0x} represents energy flowing in the positive x-direction, which is out of the volume across face (2). Similarly, the energy flowing in the y direction is $a^2cT^{0y}(y=0) - a^2cT^{0y}(y=a)$.

The sum of these rates across each face must be equal to the rate of increase of energy inside the cube $\partial/\partial t(a^3T^{00})$: This is the statement of conservation of energy. Therefore we have:

$$\begin{aligned}\frac{\partial}{\partial t}(a^3T^{00}) &= a^2c\Big[T^{0x}(x=0) - T^{0x}(x=a) + T^{0y}(y=0) - T^{0y}(y=a) \\ &+ T^{0z}(z=0) - T^{0z}(z=a)\Big] \ . \end{aligned} \tag{4.18}$$

Dividing by a^3 and taking the limit $a \to 0$ gives

$$\frac{\partial}{\partial t}T^{00} = -c\left[\frac{\partial}{\partial x}T^{0x} + \frac{\partial}{\partial y}T^{0y} + \frac{\partial}{\partial z}T^{0z}\right] \ . \tag{4.19}$$

Dividing by c we get

$$\frac{\partial}{\partial(ct)}T^{00} + \frac{\partial}{\partial x}T^{0x} + \frac{\partial}{\partial y}T^{0y} + \frac{\partial}{\partial z}T^{0z} \ . \tag{4.20}$$

Since $x^0 = ct$, $x^1 = x$, $x^2 = y$ and $x^3 = z$, we can write this as

$$T^{00}{}_{,0} + T^{01}{}_{,1} + T^{02}{}_{,2} + T^{03}{}_{,3} = 0 \ , \tag{4.21}$$

or

$$T^{0\alpha}{}_{,\alpha} = 0 \ . \tag{4.22}$$

This is the statement of the law of conservation of energy.

Similarly momentum is conserved. The same mathematics applies. with the index 0 changed to i [the spatial components] i.e.

$$T^{i\alpha}{}_{,\alpha} = 0 \ . \tag{4.23}$$

The general conservation law of energy and momentum is therefore

$$T^{\alpha\beta}{}_{,\alpha} = 0 \ . \tag{4.24}$$

This applies to any material in Special Relativity.

4.1.3 Conservation of particles

It may also happen, that in a fluid flow, the total number of particles does not change. This conservation law is derivable in the same way as the conservation of energy and momentum [EXERCISE 4.1]:

$$\frac{\partial}{\partial(ct)}N^0 + \frac{\partial}{\partial x}N^x + \frac{\partial}{\partial y}N^y + \frac{\partial}{\partial z}N^z = 0 \ , \tag{4.25}$$

or

$$N^\alpha{}_{,\alpha} = 0 \Leftrightarrow (nU^\alpha)_{,\alpha} \ . \tag{4.26}$$

We will confine ourselves to discussing fluids which obey this conservation law. This is hardly any restriction since we can always take n to be the number density of baryons.

4.1.4 Perfect fluids

Finally we come to the type of fluid which will be of most interest to us. A perfect fluid in Relativity is defined to be a fluid that has no heat conduction in the MCRF. It is the generalization of the ideal gas in ordinary thermodynamics. In the MCRF the components of $\mathbf{T}$ are

$$T^{\alpha\beta} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \ . \tag{4.27}$$

It is easy to show that in a general frame we have [EXERCISE 4.2]:

$$T^{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right) U^\alpha U^\beta + p\eta^{\alpha\beta} \ . \tag{4.28}$$

This is the energy - momentum tensor of a perfect fluid.

4.1.5 The conservation equations

We can now apply the conservation laws $T^{\alpha\beta}{}_{,\beta} = 0$ and $N^\alpha{}_{,\alpha} = 0$ to find the conservation equations. We have:

$$T^{\alpha\beta}{}_{,\beta} \ = \ \left[\left(\rho + \frac{p}{c^2}\right) U^\alpha U^\beta\right]_{,\beta} + \left(p\eta^{\alpha\beta}\right)_{,\beta} = 0$$

$$= \left[\frac{1}{n}\left(\rho + \frac{p}{c^2}\right) nU^\alpha U^\beta\right]_{,\beta} + p\eta^{\alpha\beta}{}_{,\beta} + \eta^{\alpha\beta}p_{,\beta} = 0$$

$$= nU^\beta \left[\frac{1}{n}\left(\rho + \frac{p}{c^2}\right) U^\alpha\right]_{,\beta} + \eta^{\alpha\beta}p_{,\beta} = 0 \ . \tag{4.29}$$

If we multiply by U_α and use

$$\eta^{\alpha\beta}U_\alpha = U^\beta \ , \qquad U^\alpha U_\alpha = -c^2 \ , \tag{4.30}$$

and

$$\frac{1}{2}\frac{\partial}{\partial x^\beta}(U^\alpha U_\alpha) = 0 \Leftrightarrow U^\alpha{}_{,\beta}U_\alpha = 0 \ , \tag{4.31}$$

we get

$$\frac{1}{n}\left(\rho + \frac{p}{c^2}\right) n_{,\beta}U^\beta - \rho_{,\beta}U^\beta = 0 \ , \tag{4.32}$$

which gives

$$\frac{d\rho}{d\tau} = \left(\frac{\rho + p/c^2}{n}\right)\frac{dn}{d\tau} \ , \tag{4.33}$$

where $\frac{d}{d\tau} \equiv U^\beta \frac{\partial}{\partial x^\beta}$ is the derivative along the world line of the fluid.

The i components of $T^{\alpha\beta}{}_{,\beta} = 0$ give

$$U^i{}_{,\beta}U^\beta \left(\rho + \frac{p}{c^2}\right) \eta^{i\beta}p_{,\beta} + nU^\beta U^i \left(\frac{\rho + p/c^2}{n}\right)_{,\beta} = 0 \ , \tag{4.34}$$

so in the MCRF [$U^i = 0$] this equation is

$$\frac{dU^i}{d\tau}\left(\rho + \frac{p}{c^2}\right) + \eta^{i\beta}\nabla_\beta p = 0 \ . \tag{4.35}$$

Equation (4.33) is the energy conservation equation and equation (4.35) is the momentum conservation equation [or acceleration equation].

In the Newtonian limit $v \ll c$ and $p \ll \rho c^2$ so in this case the conservations equations become [EXERCISE 4.2]

$$\frac{d\rho}{dt} = m\frac{dn}{dt} \ , \tag{4.36}$$

and

$$\rho\left[\frac{\partial \mathbf{v}}{dt} + (\mathbf{v}\cdot\nabla)\,\mathbf{v}\right] = -\nabla p \ , \tag{4.37}$$

where we have used

$$U^i{}_{,\beta}U^\beta = \frac{\partial \mathbf{v}}{dt} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} \ . \tag{4.38}$$

4.2 The Electromagnetic tensor

Maxwell's equations for the electromagnetic field [in units with $\epsilon_0 = \mu_0 = c = 1$] are:

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 4\pi \mathbf{j} \quad , \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \, ,$$
$$\nabla \cdot \mathbf{E} = 4\pi\rho \quad , \qquad \nabla \cdot \mathbf{B} = 0 \, . \tag{4.39}$$

Defining the anti-symmetric tensor $\mathbf{F}$ with components:

$$F^{\alpha\beta} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{pmatrix} \tag{4.40}$$

the electric and magnetic fields are given by

$$\mathbf{E} = (E^x, E^y, E^z) = \left(F^{01}, F^{02}, F^{03}\right) \, , \tag{4.41}$$

$$\mathbf{B} = (B^x, B^y, B^z) = \left(F^{23}, F^{31}, F^{12}\right) \, . \tag{4.42}$$

If we also define a current four-vector:

$$\mathbf{J} = (\rho, j^x, j^y, j^z) \, , \tag{4.43}$$

Maxwell's equations can be written as [EXERCISE 4.3]

$$F^{\mu\nu}{}_{,\nu} = 4\pi J^\mu \, , \tag{4.44}$$

$$F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = 0 \, , \tag{4.45}$$

where $F_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta}$. We have now expressed Maxwell's equations in tensor form as required by Special Relativity.

The first of these equations implies charge conservation

$$J^\mu{}_{,\mu} = 0 \Leftrightarrow \frac{\partial \rho}{\partial t} + \nabla . \mathbf{j} = 0 \, . \tag{4.46}$$

Proof:

$$4\pi J^\mu{}_{,\mu} = F^{\mu\nu}{}_{,\nu\mu} = F^{\nu\mu}{}_{,\nu\mu}$$

$$
\begin{aligned}
&= -F^{\mu\nu}{}_{,\nu\mu} \\
\Rightarrow F^{\mu\nu}{}_{,\nu\mu} &= 0 \\
\Rightarrow J^{\mu}{}_{,\mu} &= 0 \, . \qquad (4.47)
\end{aligned}
$$

By performing a Lorentz transformation to a frame moving with speed v in the x direction, one can calculate how the electric and magnetic fields change:

$$F^{\bar{\mu}\bar{\nu}} = \Lambda^{\bar{\mu}}{}_{\alpha}\Lambda^{\bar{\nu}}{}_{\beta}F^{\alpha\beta} \, . \qquad (4.48)$$

We find [EXERCISE 4.3] that $\mathbf{E}_{||} = E^x$ is unchanged, while

$$\mathbf{E}_{\perp} = \gamma\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \, , \qquad (4.49)$$

where $\mathbf{E}_{||}$ and $\mathbf{E}_{\perp}$ is the electric field parallel and perpendicular to $\hat{\mathbf{x}}$. Thus $\mathbf{E}$ and $\mathbf{B}$ get mixed.

The four-force on a particle of charge q and velocity $\mathbf{U}$ in an electromagnetic field is [EXERCISE 4.4]:

$$
\begin{aligned}
K^{\mu} &= qF^{\mu\nu}U_{\nu} \\
&= q\gamma\left(\mathbf{E}.\mathbf{v}, \mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \, . \qquad (4.50)
\end{aligned}
$$

The spatial part of K^{μ} is the Lorentz Force and the time part is the rate of work by this force.

By writing $\mathbf{J} = q\mathbf{U}$, Maxwell's equations give [EXERCISE 4.4]:

$$K^{\mu} = -T^{\mu\nu}{}_{,\nu} \, , \qquad (4.51)$$

where

$$T^{\mu\nu} = \tfrac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \tfrac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) \, . \qquad (4.52)$$

This is the energy momentum tensor of the electromagnetic field. Note that $T^{\mu\nu}$ is symmetric as required and the energy density is [EXERCISE 4.4]

$$T^{00} = \tfrac{1}{8\pi}\left(\mathbf{E}^2 + \mathbf{B}^2\right) \, . \qquad (4.53)$$