Introduction to algebraic geometry

Secant variety

Grassmann secant varieties
Introduction to algebraic geometry

Secant variety

Grassmann secant varieties
Definition

*Algebraic geometry* is a branch of mathematics which combines abstract algebra, especially commutative algebra, with geometry. It can be seen as the study of solution sets of systems of polynomials.
Examples in $\mathbb{R}^2$

(a) $x^2 + y^2 - 1 = 0$

(b) $(x^2 + y^2 - 1)^3 - x^2 y^3 = 0$

(c) $2x^2 y^2 - x^2 - y^2 = 0$
Example in $\mathbb{R}^3$

| diabolo | surface in $\mathbb{R}^3$ with equation $x^2 - (y^2 + z^2) = x^2(y^2 + z^2)$ |
Example in $\mathbb{R}^3$

doughnut

surface in $\mathbb{R}^3$ with equation

$$(x^2 + y^2 + z^2 + 3)^2 = 16(y^2 + z^2)$$
1. From $\mathbb{R}^N$ to $\mathbb{C}^N$

- **Why?** We want that every polynomial in one variable of degree at least one has a root. This is not the case for $\mathbb{R}$ (for example: the polynomial $x^2 + 1$ has no root in $\mathbb{R}$).

- **Solution?**
  Idea: Enlarge $\mathbb{R}$ to a set for which the property holds.
  Formally: $\mathbb{R} \subset \mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$, where $i$ is a root of the polynomial $x^2 + 1$ (hence $i^2 = -1$).
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2. From $\mathbb{C}^N$ to $\mathbb{P}^N$

- **Why?** For instance, we want that each two lines in the plane have an intersection point.

- **Solution?**
  
  Idea: $\mathbb{P}^N = \mathbb{C}^N \cup \{\text{points on infinity}\}$
  
  Formally: $\mathbb{P}^N$ is the set of elements $(x_0 : \ldots : x_N)$ with $x_0, \ldots, x_N \in \mathbb{C}$ not all zero such that

  $$(x_0 : \ldots : x_N) = (y_0 : \ldots : y_N) \iff \exists \lambda \in \mathbb{C} \setminus \{0\} : \forall i : y_i = \lambda \cdot x_i$$
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Definition
A set $X \subset \mathbb{P}^N$ is a variety if and only if $X$ is the solution set of a system of homogeneous polynomials $f_1, \ldots, f_r$ with $N + 1$ variables, i.e.

$$X = \{(x_0 : \ldots : x_N) | \forall i = 1 \ldots r : f_i(x_0, \ldots, x_N) = 0\} \subset \mathbb{P}^N.$$ 

Remarks

- The polynomial $f$ is homogeneous of degree $d$ if and only if each monomial in $f$ has total degree $d$. For example:
  
  $4x_0^3x_1^2 - 7x_0x_1x_2^3 + x_1^4x_2$ is homogeneous of degree 5.

  In this case, we have $f(\lambda \cdot x_0, \ldots, \lambda \cdot x_N) = \lambda^d \cdot f(x_0, \ldots, x_N)$ for each $\lambda \in \mathbb{C}$, hence $X$ is "well-defined".
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Grassmann secant varieties and plane curves with total inflection points

Introduction to algebraic geometry

Algebraic variety

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We will always assume irreducibility.

Every variety has a dimension, usually denoted by $n$. A curve is a variety of dimension 1, a surface is a variety of dimension 2, etc.

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Definition
Let $r$ and $n$ be non-zero integers. The $r$-uple Veronese embedding of $\mathbb{P}^n$ is the image of the map $\nu_r : \mathbb{P}^n \to \mathbb{P}^N$ with $N = \binom{n+r}{r} - 1$, sending $(x_0 : \ldots : x_n)$ to all possible monomials of degree $r$.

Special cases

▶ The variety $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ is called the Veronese surface. Note that

$$\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5 : (x_0 : x_1 : x_2) \mapsto (x_0^2 : x_1^2 : x_2^2 : x_1x_2 : x_0x_2 : x_0x_1).$$

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Definition
The Grassmannian $\mathbb{G}(h, N)$ is the set of $h$-dimensional linear subspaces in $\mathbb{P}^N$.

Property
The Grassmannian $\mathbb{G}(h, N)$ is a variety of dimension $(N - h)(h + 1)$ and it can be embedded in a large projective space using the so-called Grassmann embedding.
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Grassmann secant varieties
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Let $X \subset \mathbb{P}^N$ be a non-degenerate $n$-dimensional variety.

- A line passing through two different points $P_0$ and $P_1$ of $X$ is called a secant line of $X$. (Notation: $\langle P_0, P_1 \rangle$)
- The secant variety $S(X) \subset \mathbb{P}^N$ of $X$ is the closure of the union of all secant lines of $X$.

Remark
$\dim S(X) \leq \min\{2n + 1, N\}$
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Grassmann secant varieties and plane curves with total inflection points

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We say that the variety $X \subset \mathbb{P}^N$ is defective if and only if
\[ \dim S(X) < \min\{2n + 1, N\}. \]

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If $2n + 1 \leq N$, the variety $X \subset \mathbb{P}^N$ is defective if and only if a
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Grassmann secant varieties and plane curves with total inflection points

- Secant variety
- Connection with projections

Projection of a variety

- variety \( X \subset \mathbb{P}^N \)
- point \( P \in \mathbb{P}^N \setminus X \)
- hyperplane \( \mathbb{P}^{N-1} \subset \mathbb{P}^N \) with \( P \notin \mathbb{P}^{N-1} \)
- projection \( p : X \to \mathbb{P}^{N-1} \) from the point \( P \)
- for example: if \( P_0 \in X \), \( p(P_0) = \langle P, P_0 \rangle \cap \mathbb{P}^{N-1} \)
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Property

Let $X \subset \mathbb{P}^N$ be a smooth variety. The projection $p : X \rightarrow p(X)$ from $P$ is an isomorphism if and only if $P \not\in S(X)$.

Corollary

Any smooth variety $X$ of dimension $n$ admits an embedding in a projective space $\mathbb{P}^{2n+1}$. We can go further and project $X$ isomorphically in some $\mathbb{P}^m$ with $m < 2n + 1$ if $\dim S(X) < 2n + 1$. 
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We can go further and project $X$ isomorphically in some $\mathbb{P}^m$ with $m < 2n + 1$ if $\dim S(X) < 2n + 1$. 
Example: The Veronese surface is defective

Note that $\nu_2 : \mathbb{P}^2 \rightarrow \mathbb{P}^5 : (x_0 : x_1 : x_2) \mapsto (x_0^2 : x_1^2 : x_2^2 : x_1x_2 : x_0x_2 : x_0x_1)$. 
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Example: A cone over a curve is defective

Let $X \subset \mathbb{P}^N$ be a cone over the curve $C$ with vertex $T$, i.e.

$$X = \bigcup_{Q \in C} \langle T, Q \rangle.$$

Assume $N \geq 5$, hence the expected dimension of $S(X)$ is $\min\{2n + 1, N\} = 5$.

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Theorem [F. Severi, 1901]
The only defective surfaces are the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$
and cones over a curve.
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Let $X \subset \mathbb{P}^N$ be a non-degenerate variety and let $h$ and $k$ be integers such that $h \leq k$. The $(h, k)$-Grassmann secant variety $G_{h,k}(X) \subset \mathbb{G}(h, N)$ of $X$ is the closure of the set of $h$-dimensional subspaces contained in the span of $k + 1$ independent points of $X$.

Special cases

$\blacktriangleright \quad G_{0,1}(X) = S(X)$

$\blacktriangleright \quad G_{0,k}(X) \subset \mathbb{P}^N$

Remark
$\dim(G_{h,k}(X)) \leq \min\{(k + 1)n + (k - h)(h + 1), (N - h)(h + 1)\}$
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Remark

$\dim(G_{h,k}(X)) \leq \min\{(k + 1)n + (k - h)(h + 1), (N - h)(h + 1)\}$
Definition
Let $X \subset \mathbb{P}^N$ be a non-degenerate variety and let $h$ and $k$ be integers such that $h \leq k$. The $(h, k)$-Grassmann secant variety $G_{h,k}(X) \subset \mathbb{G}(h, N)$ of $X$ is the closure of the set of $h$-dimensional subspaces contained in the span of $k + 1$ independent points of $X$.

Special cases
- $G_{0,1}(X) = S(X)$
- $G_{0,k}(X) \subset \mathbb{P}^N$

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$\dim(G_{h,k}(X)) \leq \min\{(k + 1)n + (k - h)(h + 1), (N - h)(h + 1)\}$
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The difference $\delta_{h,k}(X)$ between the expected and the actual dimension of $G_{h,k}(X)$ is called the $(h, k)$-defect of $X$.

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Question (Edward Waring, 1770):

Is is possible to write each integer as the sum of a fixed number $k(r)$ of $r$th powers of integers, where $r \in \mathbb{N} \setminus \{0\}$ and $k(r)$ only depends on $r$?

Answer: YES for every $r$ (Hilbert, 1909)

Examples:

- $k(2) = 4$ ($2007 = 43^2 + 11^2 + 6^2 + 1^2$)
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First generalization

Given positive integers $k, n, r$, is it possible to write a general homogeneous form $f \in \mathbb{C}[x_0, \ldots, x_n]_r$ as

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for some linear forms $\ell_0, \ldots, \ell_k \in \mathbb{C}[x_0, \ldots, x_n]_1$?

Connection with Grassmann secant varieties

Answer to the above question is 'YES' $\iff G_{0,k}(\nu_r(\mathbb{P}^n)) = \mathbb{P}^N$.

Solution of this problem

J. Alexander and A. Hirschowitz, 1995
Grassmann secant varieties and plane curves with total inflection points

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Given positive integers $h, k, n, r$, is it possible to write general homogeneous forms $f_0, \ldots, f_h \in \mathbb{C}[x_0, \ldots, x_n]_r$ as linear combinations of $r$-th powers of the same linear forms $\ell_0, \ldots, \ell_k \in \mathbb{C}[x_0, \ldots, x_n]_1$?

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Grassmann secant varieties and plane curves with total inflection points

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Answer to the above question is 'YES' $\iff G_{h,k}(\nu_r(\mathbb{P}^n)) = \mathcal{G}(h, N)$.
Some results for the case $h > 0$

- classification of $(1, 2)$-defective surfaces [Chiantini and Coppens, 2001]
- curves are not $(h, k)$-defective [Chiantini and Ciliberto, 2002]
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Theorem 1 [Ciliberto and Cools]

If $X \subset \mathbb{P}^N$ is a non-degenerate $n$-dimensional variety such that $G_{h,k}(X) \neq \mathbb{G}(h, N)$, we have

$$\dim(G_{h,k}(X)) \geq (k + 1)n + (k - h)(h + 1) - (k - h)(n - 1). \quad (*)$$

**Definition**

We say that the variety $X \subset \mathbb{P}^N$ is $(h, k)$-extremal if and only if $G_{h,k}(X) \neq \mathbb{G}(h, N)$ and the equality in $(*)$ is attained.
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Theorem 2 [Ciliberto and Cools]

Let $X \subset \mathbb{P}^N$ be a non-degenerate, $n$–dimensional, $(h, k)$-extremal variety for integers $0 \leq h \leq k$. Then either $h = k$ and $N > n + k$, or $n = 1$ and $N > k + \frac{k+1}{h+1}$, or we have one of the following possibilities:

1. $N > n + k + \frac{k-h}{h+1}$ and $X \subset \mathbb{P}^N$ is a cone over a curve with vertex a linear subspace $\mathbb{P}^{n-2}$;

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Let $X^n$ be a non-degenerate variety in $\mathbb{P}^N$ and let $k \geq n$ be an integer. If $X$ is $(k - 1, k)$-defective with defect $\delta_{k-1,k}(X) = a > 0$, we have $a \leq n - 1$ and one of the following two properties hold for $k + 1$ general points $P_0, \ldots, P_k$ on $X$:

1. For each $i \in \{0, \ldots, k\}$, there exists a linear subspace $\gamma_i$ of dimension $a$ on $X$ containing $P_i$ so that $\dim \langle \gamma_0, \ldots, \gamma_k \rangle = k + a$.

2. There exists an $a$-dimensional rational normal scroll $\gamma$ of degree $k + 1$ on $X$ containing $P_0, \ldots, P_k$ ($\dim \langle \gamma \rangle = k + a$).

Furthermore, if $X$ satisfies one of the two properties, $X$ is $(k - 1, k)$-defective with defect at least $a$. 
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Grassmann secant varieties and plane curves with total inflection points

Grassmann secant varieties

Case $h = k - 1$

Example: $X = \nu_2(\mathbb{P}^3) \subset \mathbb{P}^9$ is $(4, 5)$-defective

- $P_0, \ldots, P_5$ general points on $X$
- $Q_0, \ldots, Q_5$ points in $\mathbb{P}^3$ such that $\nu_2(Q_i) = P_i$
- there exists a rational normal curve $C \subset \mathbb{P}^3$ of degree 3 through $Q_0, \ldots, Q_5$
- the image of $C$ under $\nu_2$ is a rational normal curve of degree 6 in $X$ through $P_0, \ldots, P_6$
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Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate $n$-dimensional $(k-1, k)$-defective variety with defect $\delta_{k-1, k}(X) = a \geq n - 2$ for an integer $k \geq n$. Then $X$ is one of the following varieties:

1. $X^n$ is a rational normal scroll of minimal degree $k + 2$ in $\mathbb{P}^{n+k+1}$ ($a = n - 1$);
2. $n \geq 3$ and $X^n$ is a rational normal scroll of minimal degree $k + 3$ in $\mathbb{P}^{n+k+2}$ ($a = n - 2$);
3. $n \geq 3$ and $X^n$ is the projection in $\mathbb{P}^{n+k+1}$ of a $n$-dimensional rational normal scroll of minimal degree $k + 3$ in $\mathbb{P}^{n+k+2}$ ($a = n - 2$);
4. $n = k = 3$ and $X^3$ is a hyperplane section of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^8$ ($a = 1$);
5. $n = 3$, $k = 4$ and $X^3$ is a (linearly normal) embedding of the blowing-up of $\mathbb{P}^3$ in a point in $\mathbb{P}^8$ ($a = 1$);
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3. $n \geq 3$ and $X^n$ is the projection in $\mathbb{P}^{n+k+1}$ of a $n$-dimensional rational normal scroll of minimal degree $k + 3$ in $\mathbb{P}^{n+k+2}$ ($a = n - 2$);

4. $n = k = 3$ and $X^3$ is a hyperplane section of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^8$ ($a = 1$);

5. $n = 3$, $k = 4$ and $X^3$ is a (linearly normal) embedding of the blowing-up of $\mathbb{P}^3$ in a point in $\mathbb{P}^8$ ($a = 1$);

6. $n = 3$, $k = 5$ and $X^3 = \nu_2(\mathbb{P}^3) \subset \mathbb{P}^9$ ($a = 1$).
Theorem 4 [Cools]

Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate $n$-dimensional $(k-1,k)$-defective variety with defect $\delta_{k-1,k}(X) = a \geq n - 2$ for an integer $k \geq n$. Then $X$ is one of the following varieties:

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Some results for the case $h > 0$

- classification of $(1, 2)$-defective surfaces [Chiantini and Coppens, 2001]
- curves are not $(h, k)$-defective [Chiantini and Ciliberto, 2002]
- classification of smooth $(2, 3)$-defective threefolds [Coppens, 2004]
- classification of $(1, k)$-defective surfaces [Chiantini and Ciliberto, 2005]
- classification of $(h, k)$-extremal varieties [Ciliberto and Cools]
- classification of smooth $(k - 1, k)$-defective surfaces and threefolds (if $k \geq 3$) [Cools]
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