A Tropical Proof of the Brill-Noether Theorem
(joint work with J. Draisma, S. Payne and E. Robeva)

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Classical Brill-Noether Theory (BNT)

Fix integers $g, r, d \geq 0$, and denote the Brill-Noether number by

$$\rho(g, r, d) := g - (r + 1)(g - d + r).$$

(a) If $\rho(g, r, d) \geq 0$, then every smooth curve $X/\mathbb{C}$ of genus $g$
has a divisor $D$ of degree $\leq d$ and rank $r$.

(b) If $\rho(g, r, d) < 0$, then on a general smooth curve $X/\mathbb{C}$ of
genus $g$, there is no divisor $D$ of degree $\leq d$ and rank $r$. 
BNT for metric graphs?

In the paper “Specialization of linear systems from curves to graphs”, M. Baker proves part (a) of BNT for metric graphs, i.e.

If \( \rho(g, r, d) := g - (r + 1)(g - d + r) \geq 0 \), then every metric graph \( \Gamma \) of genus \( g \) has a divisor \( D \) of degree \( \leq d \) and rank \( r \).

He also conjectures that part (b) of BNT is true under the following form.

If \( \rho(g, r, d) < 0 \), then there exists a metric graph \( \Gamma \) of genus \( g \) for which there is no divisor \( D \) of degree \( \leq d \) and rank \( r \).
Consider the metric graph $\Gamma$ of genus $g$ which is a chain of $g$ loops, where the lengths of the two edges of the $i$th loop are equal to $\ell_i$ and $m_i$. 

$\ell_1 \quad \ell_g$

$\ell_1 \quad \ell_g$

$v_0 \quad v_1 \quad v_{g-1} \quad v_g$

$m_1 \quad m_g$
Example: \( g = 3 \) and \( \ell_i = m_i = 1 \) for all \( i \)

\[
\begin{align*}
    &v_0 \quad v_1 \quad v_2 \quad v_3 \\
    &v_0 \quad v_1 \quad v_2 \quad v_3
\end{align*}
\]

is equivalent with

\[
\begin{align*}
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    &v_0 \quad v_1 \quad v_2 \quad v_3
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is equivalent with

\[
\begin{align*}
&v_0 \quad 2 \\
&v_1 \\
&v_2 \\
&v_3 \quad 2
\end{align*}
\]
Example: \( g = 3 \) and \( \ell_i = m_i = 1 \) for all \( i \)

The divisor \( D = 2v_0 \) has rank 1. So for \( g = 3, \ d = 2 \) and \( r = 1 \), there does exists a divisor on \( \Gamma \) of rank \( r \) and degree \( \leq d \), but
\[
\rho = g - (r + 1)(g - d + r) = -1.
\]
Conclusion: not a good example for part (b) of BNT for metric graphs.
Example: $g = 3$ and $\ell_i = m_i = 1$ for all $i$

\[ \begin{array}{c}
\bullet v_0 \bullet v_1 \bullet v_2 \bullet v_3 \\
2 \end{array} \]

The divisor $D = 2v_0$ has rank 1. So for $g = 3$, $d = 2$ and $r = 1$, there does exists a divisor on $\Gamma$ of rank $r$ and degree $\leq d$, but

\[ \rho = g - (r + 1)(g - d + r) = -1. \]

Conclusion: not a good example for part (b) of BNT for metric graphs.
Idea

Assume that the edge lengths $\ell_i$ and $m_i$ are generic positive numbers.

Precise definition: The metric graph $\Gamma$ is *generic* if none of the ratios $\ell_i/m_i$ is equal to the ratio of two positive integers whose sum is less or equal to $2g - 2$.

Example: $\ell_i = 2g - 2$ and $m_i = 1$ for all $i$. 
Theorem
Assume $\Gamma$ is generic.

(a) If $\rho(g, r, d) < 0$, there is no divisor $D$ on $\Gamma$ with degree $\leq d$ and rank $r$. (i.e. part (b) of BNT for metric graphs)

(b) If $\rho(g, r, d) \geq 0$, then the dimension of $W^r_d(\Gamma)$ is equal to $\min\{\rho(g, r, d), g\}$.

(c) If $\rho(g, r, d) = 0$, there are exactly

$$
\lambda = g! \prod_{i=0}^{r} \frac{i!}{(g - d + r + i)!}
$$

linear systems on $\Gamma$ of rank $r$ and degree $d$. 
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(c) If $\rho(g, r, d) = 0$, there are exactly

$$\begin{align*}
\lambda &= g! \prod_{i=0}^{r} \frac{i!}{(g - d + r + i)!} \\
&= g! \prod_{i=0}^{r} \frac{i!}{(g - d + r + i)!}
\end{align*}$$

linear systems on $\Gamma$ of rank $r$ and degree $d$. 

Proposition
Assume $\Gamma$ is generic, $r = 1$ and $g = 2d - 2$ (so $\rho(g, r, d) = 0$). Then there is a bijection between $W^1_d(\Gamma)$ and

$$\left\{ \begin{array}{l}
\text{lattice paths } p = (p_0, \ldots, p_g) \text{ in } \mathbb{Z} \text{ satisfying } p_0 = p_g = 1, \\
p_i \geq 1 \text{ and } p_i - p_{i-1} = \pm 1 \text{ for all } i \in \{1, \ldots, g\}
\end{array} \right\}$$

as follows. If $p = (p_0, \ldots, p_g)$ is such a path, let $D_p$ be the divisor on $\Gamma$ with

- one chip in $v_0$,
- one (extra) chip on the unique point $w_i$ of the $i$th loop satisfying $p_{i-1}v_{i-1} + w_i \sim p_i v_i$ if $p_i - p_{i-1} = 1$,
- no (extra) chips on the $i$th loop if $p_i - p_{i-1} = -1$.

Note that $D_p$ is $v_0$-reduced. The bijection maps $p$ to the linear system $|D_p|$. 
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as follows. If $p = (p_0, \ldots, p_g)$ is such a path, let $D_p$ be the divisor on $\Gamma$ with

- one chip in $v_0$,
- one (extra) chip on the unique point $w_i$ of the $i$th loop satisfying $p_{i-1}v_{i-1} + w_i \sim p_i v_i$ if $p_i - p_{i-1} = 1$,
- no (extra) chips on the $i$th loop if $p_i - p_{i-1} = -1$.

Note that $D_p$ is $v_0$-reduced. The bijection maps $p$ to the linear system $|D_p|$. 
Proposition

Assume $\Gamma$ is generic, $r = 1$ and $g = 2d - 2$ (so $\rho(g, r, d) = 0$). Then there is a bijection between $W_d^1(\Gamma)$ and

\[
\left\{ \text{lattice paths } p = (p_0, \ldots, p_g) \text{ in } \mathbb{Z} \text{ satisfying } p_0 = p_g = 1, \right. \\
\left. p_i \geq 1 \text{ and } p_i - p_{i-1} = \pm 1 \text{ for all } i \in \{1, \ldots, g\} \right\}
\]

as follows. If $p = (p_0, \ldots, p_g)$ is such a path, let $D_p$ be the divisor on $\Gamma$ with

- one chip in $v_0$,
- one (extra) chip on the unique point $w_i$ of the $i$th loop satisfying $p_i - 1 \sim v_i - v_{i-1}$ if $p_{i} - p_{i-1} = 1$,
- no (extra) chips on the $i$th loop if $p_i - p_{i-1} = -1$.

Note that $D_p$ is $v_0$-reduced. The bijection maps $p$ to the linear system $|D_p|$. 

Example: $r = 1, g = 6, d = 4$

In this case, there are precisely five lattice paths that satisfy the conditions:

- $(1, 2, 3, 4, 3, 2, 1)$
- $(1, 2, 3, 2, 3, 2, 1)$
- $(1, 2, 3, 2, 1, 2, 1)$
- $(1, 2, 1, 2, 3, 2, 1)$
- $(1, 2, 1, 2, 1, 2, 1)$
Example: \( r = 1, \ g = 6, \ d = 4, \ p = (1, 2, 3, 4, 3, 2, 1) \)

The divisor \( D_p \) is equal to

since we have e.g.

- one chip at \( v_0 \),
- one chip at the unique point \( w_2 \) of the 2nd loop satisfying \( p_1 v_1 + w_2 = 2v_1 + w_2 \sim p_2 v_2 = 3v_2 \) since \( p_2 - p_1 = 1 \),
- no chip at the 4th loop since \( p_4 - p_3 = -1 \).
Example: \( r = 1, \ g = 6, \ d = 4, \ p = (1, 2, 3, 4, 3, 2, 1) \)

The divisor \( D_p \) is equal to

\[
\begin{array}{cccccccc}
9 & 8 & 7 & 10 & 10 & 10 & 10 & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & \\
\end{array}
\]

and is equivalent with

\[
\begin{array}{cccccccc}
10 & 8 & 7 & 10 & 10 & 10 & 10 & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & \\
\end{array}
\]
Example: $r = 1$, $g = 6$, $d = 4$, $p = (1, 2, 3, 4, 3, 2, 1)$

The divisor $D_p$ is equal to

\[ v_0 \cdot 1 \quad v_1 \cdot 1 \quad v_2 \cdot 1 \quad v_3 \cdot 1 \quad v_4 \cdot 1 \quad v_5 \cdot 1 \quad v_6 \cdot 1 \]

and is equivalent with

\[ v_0 \cdot 1 \quad v_1 \cdot 1 \quad v_2 \cdot 3 \quad v_3 \cdot 1 \quad v_4 \cdot 1 \quad v_5 \cdot 1 \quad v_6 \cdot 1 \]
Example: $r = 1$, $g = 6$, $d = 4$, $p = (1, 2, 3, 4, 3, 2, 1)$

The divisor $D_p$ is equal to

and is equivalent with
Example: $r = 1$, $g = 6$, $d = 4$, $p = (1, 2, 3, 4, 3, 2, 1)$

The divisor $D_p$ is equal to

$$v_0 v_1 v_2 v_3 v_4 v_5 v_6$$

$$1 1 1 1 1 1$$

$$10 10 10 10 7 3 1$$

and is equivalent with

$$v_0 v_1 v_2 v_3 v_4 v_5 v_6$$

$$1 1 1 1 1 1$$

$$10 10 10 3 1 10 10 3$$
Example: \( r = 1, \, g = 6, \, d = 4, \, p = (1, 2, 3, 4, 3, 2, 1) \)

The divisor \( D_p \) is equal to

\[
\begin{array}{cccccccc}
\text{ } & v_0 & 1 & v_1 & 1 & v_2 & 1 & v_3 & 1 \\
1 & & & & & & & & \\
\end{array}
\]

and is equivalent with

\[
\begin{array}{cccccccc}
\text{ } & v_0 & & v_1 & & v_2 & & v_3 & \\
1 & & & & & & & & \\
\end{array}
\]
Example: $r = 1, g = 6, d = 4, p = (1, 2, 3, 4, 3, 2, 1)$

The divisor $D_p$ is equal to

and is equivalent with

\[
\begin{array}{cccccccccccc}
\text{v}_0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \text{v}_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \text{v}_2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \text{v}_3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \text{v}_4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \text{v}_5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \text{v}_6 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 &
\end{array}
\]
Corollary of the theorem

Using Baker’s Specialization Lemma, we get a new (tropical) characteristic-free proof of the BNT for curves.
Main result of the recent preprint "Linear pencils on graphs and on real curves" (joint work with M. Coppens)

There exists a smooth complex curve \( X \) (resp. \( X' \)) of genus \( g = 2d - 2 \) defined over \( \mathbb{R} \) having exactly \( \lambda = \frac{1}{d} \binom{2d-2}{d-1} \) linear pencils of degree \( d \) such that all of them (resp. exactly \( \lambda' = \binom{d-1}{\left\lfloor \frac{d-1}{2} \right\rfloor} \) of them) are real.

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
 d & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
 g & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\
\hline
 \lambda & 1 & 2 & 5 & 14 & 42 & 132 & 429 & 1430 & 4862 \\
\hline
 \lambda' & 1 & 2 & 3 & 6 & 10 & 20 & 35 & 70 & 126 \\
\hline
\end{array}
\]

Note that \( \lim_{d \to \infty} \frac{\lambda'}{\lambda} = 0 \).